PHYS 301 - Assignment #1 sol'ns - 20240920

1.

Divergence theorem:

$$\int_V \vec{\nabla}\cdot\vec{v}\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$$ 0.5

$\vec{v} = 2r^3\,\hat{r}$

volume: sphere of radius R.

Volume Integral:

In spherical coords. 0.5 $d\tau = r^2 \sin\theta\, d\theta\, d\phi\, dr$

Since $\vec{v}$ has only an r-component (v_r), need only the r-component of $\vec{\nabla}\cdot\vec{v}$

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)$$ 0.5

$$\therefore \vec{\nabla}\cdot\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2(2r^3)\right) = \frac{2}{r^2}\frac{\partial}{\partial r}\left(r^5\right)$$

$$= \frac{10}{r^2}r^4 = \underline{\underline{10r^2}} \quad 0.5$$

$$\therefore \int \vec{\nabla}\cdot\vec{V}\,d\tau = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{R} 10r^2\, r^2\sin\theta\, d\theta\, d\phi\, dr$$

$$= 10\underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi}\underbrace{\int_{\theta=0}^{\pi}\sin\theta\, d\theta}_{2}\underbrace{\int_{r=0}^{R} r^4\, dr}_{\frac{R^5}{5}} = \boxed{8\pi R^5}$$

1

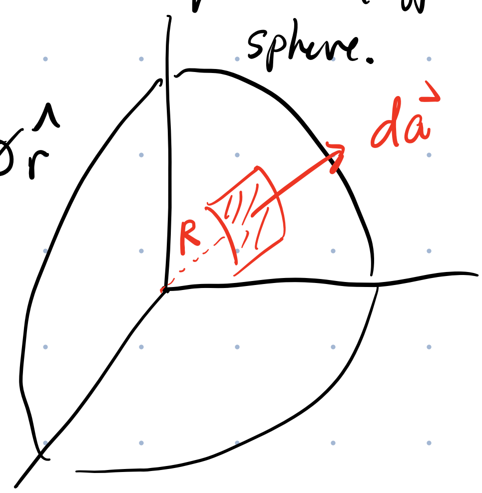
Surface Integral:

0.5

$$d\vec{a} = R^2\sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$\oint \vec{V}\cdot d\vec{a}$$

1st quadrant of sphere.



since $r = R$ on the sphere's surface

$$\vec{V}\cdot d\vec{a} = \left(2R^3\hat{r}\right)\cdot\left(R^2\sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$= 2R^5\sin\theta\, d\theta\, d\phi \quad 0.5$$

$$\therefore \oint \vec{V}\cdot d\vec{a} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} 2R^5 \sin\theta \, d\theta \, d\phi$$

$$= 2R^5 \underbrace{\int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin\theta \, d\theta}_{4\pi}$$

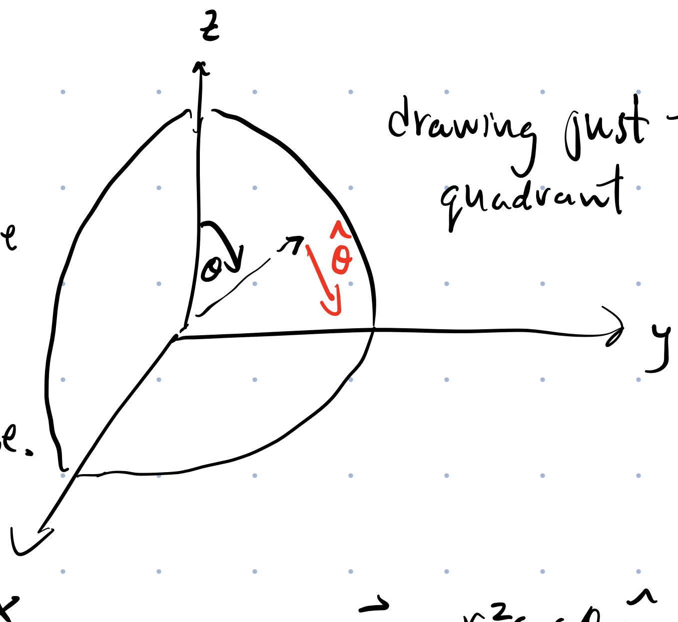
$$= \boxed{8\pi R^5}$$

1 same as before. ✓

2. Divergence Theorem:

$$\int_V \vec{\nabla}\cdot\vec{v}\,d\tau = \oint_S \vec{v}\cdot d\vec{a}$$

volume/surface is a hemisphere of radius R.



drawing just the first quadrant

Two surfaces:

0.5 ① curved surface

w/ $d\vec{a} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$

② flat btm surface.

In spherical coords, xy-plane corresponds to $\theta = \pi/2$ & $0 < \phi < 2\pi$,

$0 < r < R$.

$\therefore d\vec{a} = r \underbrace{\sin\theta}_{1}\, dr\, d\phi\, \hat{\theta}$

becomes $d\vec{a} = r\,dr\,d\phi\, \hat{\theta}$ 0.5

$$\vec{v} = r^2\cos\theta\,\hat{r} + r^2\cos\phi\,\hat{\theta} - r^2\cos\theta\sin\phi\,\hat{\phi}$$

$$d\tau = r^2\sin\theta\, d\theta\, d\phi\, dr$$

0.5

volume integral

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$$

0.5

$$\frac{\partial}{\partial r}\left(r^2 v_r\right) = \frac{\partial}{\partial r}\left(r^4\cos\theta\right) = 4r^3\cos\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\, v_\theta\right) = \frac{\partial}{\partial\theta}\left(\sin\theta\, r^2\cos\phi\right) = r^2\cos\theta\cos\phi$$

$$\frac{\partial v_\phi}{\partial\phi} = \frac{\partial}{\partial\phi}\left(-r^2\cos\theta\sin\phi\right) = -r^2\cos\theta\cos\phi$$

$$\therefore \vec{\nabla}\cdot\vec{v} = 4r\cos\theta + r\cancel{\frac{\cos\phi}{\tan\theta}} - r\cancel{\frac{\cos\phi}{\tan\theta}}$$

$$= 4r\cos\theta$$

hemisphere!

$$\therefore \int \vec{\nabla}\cdot\vec{v}\, d\tau = 4\underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi}\ \underbrace{\int_{\theta=0}^{\pi/2}\cos\theta\sin\theta\, d\theta}_{1/2}\ \underbrace{\int_{r=0}^{R} r^3\, dr}_{\frac{R^4}{4}}$$

$$\int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta \qquad u = \cos\theta \begin{cases} \theta = 0, & u = 1 \\ \theta = \pi/2, & u = 0 \end{cases}$$

$$du = -\sin\theta \, d\theta \Rightarrow \sin\theta \, d\theta = -du$$

$$= \int_{u=1}^{0} -u \, du = \int_{u=0}^{1} u \, du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\therefore \int \vec{v}\cdot\vec{v} \, d\tau = \cancel{4}(\cancel{2}\pi)\left(\frac{1}{\cancel{4}}\right)\left(\frac{R^4}{\cancel{4}}\right) = \boxed{\pi R^4}$$

1

<u>Surface Integral</u>

$$\oint \vec{v}\cdot d\vec{a} = \int_{top} \vec{v}\cdot d\vec{a} + \int_{btm} \vec{v}\cdot d\vec{a}$$

0.5

Top. $d\vec{a} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

$$\therefore \vec{v}\cdot d\vec{a} = R^4 \cos\theta \sin\theta \, d\theta \, d\phi$$

$$\int_{top} \vec{v}\cdot d\vec{a} = R^4 \underbrace{\int_{\phi=0}^{2\pi} d\phi}_{2\pi} \underbrace{\int_{\theta=0}^{\pi/2} \cos\theta \sin\theta \, d\theta}_{\frac{1}{2} \text{ (as above)}}$$

← hemisphere

$= \pi R^4.$ 0.5

Btm. $d\vec{a} = r\,dr\,d\phi\,\hat{\theta}$

$\therefore \vec{V}\cdot d\vec{a} = r^3 \cos\phi\, dr\, d\phi$

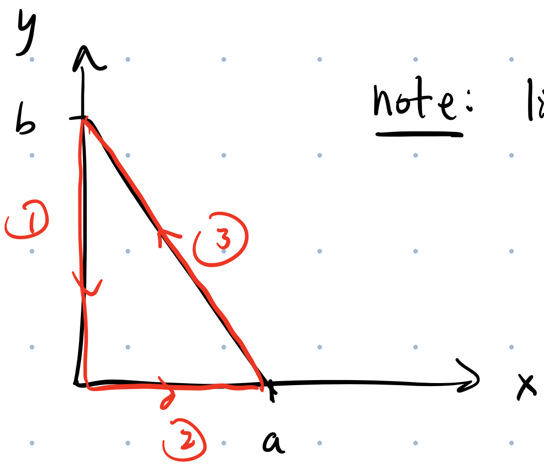
$$\int_{btm} \vec{V}\cdot d\vec{a} = \underbrace{\int_{\phi=0}^{2\pi} \cos\phi\, d\phi}_{\sin\phi \Big|_0^{2\pi} = 0} \underbrace{\int_{r=0}^{R} r^3\, dr}_{\frac{R^4}{4}} = \underline{\underline{0}}.$$ 0.5

$$\therefore \oint \vec{V}\cdot d\vec{a} = \pi R^4 + 0 = \boxed{\pi R^4}$$ 0.5

same as before ✓

3. Stoke's Theorem: $\int_S \vec{\nabla} \times \vec{V} \cdot d\vec{a} = \oint_P \vec{V} \cdot d\vec{l}$

Surface / path



note: line labelled 3 given by:

$$y = -\frac{b}{a}x + b = b\left(1 - \frac{x}{a}\right)$$

0.5

Surface Integral:

by RHR, $d\vec{a}$ is out of screen/page.

$$d\vec{a} = dx\,dy\,\hat{z}$$

$\therefore$ $0 < x < a$ 0.5

$0 < y < b\left(1 - \frac{x}{a}\right)$

$\vec{V} = ay\hat{x} + bx\hat{y}$ $\therefore$ $\vec{\nabla} \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ay & bx & 0 \end{vmatrix}$

$$= \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(b-a)$$

$$\therefore \vec{\nabla} \times \vec{V} = (b-a)\hat{z} \quad 0.5$$

$$\vec{\nabla} \times \vec{V} \cdot d\vec{a} = (b-a)\,dx\,dy$$

$$\int \vec{\nabla} \times \vec{V} \cdot d\vec{a} = \int_{x=0}^{a} \int_{y=0}^{b\left(1-\frac{x}{a}\right)} (b-a)\,dy\,dx$$

$$= (b-a)\int_{x=0}^{a} b\left(1-\frac{x}{a}\right)dx$$

$$= b(b-a)\left[x - \frac{x^2}{2a}\right]\Bigg|_0^a$$

$$= b(b-a)\underbrace{\left(a - \frac{a}{2}\right)}_{a/2}$$

$$\boxed{\therefore \int \vec{\nabla} \times \vec{V} \cdot d\vec{a} = \frac{ab}{2}(b-a)} \quad 1$$

Line Intergal

$$\oint_P \vec{V}\cdot d\vec{l} = \int_{①} \vec{V}\cdot d\vec{l} + \int_{②} \vec{V}\cdot d\vec{l} + \int_{③} \vec{V}\cdot d\vec{l}$$ 0.5

①: $d\vec{l} = dy\,\hat{y}$ w/ $y: b \to 0$ & $x = 0$

$$\vec{V}\cdot d\vec{l} = -b\cancel{x}^{\,0}\,dy = 0$$

$$\int_{①} \vec{V}\cdot d\vec{l} = 0$$ 0.25

②: $d\vec{l} = dx\,\hat{x}$ w/ $x: 0 \to a$ & $y = 0$

$$\therefore \vec{V}\cdot d\vec{l} = a\cancel{y}^{\,0}\,dx = 0$$

$$\therefore \int_{②} \vec{V}\cdot d\vec{l} = 0$$ 0.25

③ $d\vec{l} = dx\,\hat{x} + dy\,\hat{y}$

$$y = b\left(1-\frac{x}{a}\right) \quad \therefore dy = -\frac{b}{a}dx$$

$$\therefore d\vec{l} = dx\,\hat{x} - \frac{b}{a}dx\,\hat{y}$$

$$= \left(\hat{x} - \frac{b}{a}\hat{y}\right)dx \quad \& \quad x: a \to 0$$

$$\therefore \vec{V}\cdot d\vec{l} = \left(ay - \frac{b^2}{a}x\right)dx$$

but, again, $y = b\left(1-\frac{x}{a}\right)$

$$\therefore \vec{V}\cdot d\vec{l} = \left[ab\left(1-\frac{x}{a}\right) - \frac{b^2}{a}x\right]dx$$

$$= \left[ab - bx - \frac{b^2}{a}x\right]dx$$

$$= b\left[a - x\left(1+\frac{b}{a}\right)\right]dx$$

0.5

$$\therefore \int_{③} \vec{V}\cdot d\vec{l} = b\int_{x=a}^{0}\left[a - x\left(1+\frac{b}{a}\right)\right]dx$$

$$= b\left[ax - \frac{x^2}{2}\left(1+\frac{b}{a}\right)\right]\Bigg|_a^0$$

$$= b\left[a(0-a) - \frac{1}{2}(0-a^2)\left(1+\frac{b}{a}\right)\right]$$

$$= b\left[-a^2 + \frac{a^2}{2}\left(1+\frac{b}{a}\right)\right]$$

$$= \frac{ab}{2}\left[-2a + a\left(1+\frac{b}{a}\right)\right]$$

$$= \frac{ab}{2}\left[-2a + a + b\right]$$

$$\therefore \int_{③} \vec{V}\cdot d\vec{l} = \frac{ab}{2}(b-a)$$ 0.5

0.5

$$\therefore \oint \vec{V}\cdot d\vec{l} = \underbrace{\int_{①} \vec{V}\cdot d\vec{l}}_{0} + \underbrace{\int_{②} \vec{V}\cdot d\vec{l}}_{0} + \underbrace{\int_{③} \vec{V}\cdot d\vec{l}}_{\frac{ab}{2}(b-a)} = \boxed{\frac{ab}{2}(b-a)}$$

same as before ✓

4. (a) Show that $\int \delta(kx)dx = \int \frac{1}{|k|}\delta(x)dx$

s.t. $\delta(kx) = \frac{1}{|k|}\delta(x)$

Start w/ $\int_{-\varepsilon}^{\varepsilon} \delta(kx)dx$ & make the sub. $u = kx$

$\therefore du = k\,dx$ 0.5

or $dx = \frac{du}{k}$ 0.5

$\therefore \int_{-k\varepsilon}^{k\varepsilon} \delta(u)\frac{du}{k}$

choose integration limits

$-\varepsilon < x < \varepsilon$

s.t. it contains $x=0$.

when $x=\varepsilon$, $u=k\varepsilon$

$x=-\varepsilon$, $u=-k\varepsilon$

Case 1: $k>0$.

$$\int_{-\varepsilon}^{\varepsilon} \delta(kx)dx = \frac{1}{k}\underbrace{\int_{-k\varepsilon}^{k\varepsilon}\delta(u)du}_{1\,=\,\int_{-\varepsilon}^{\varepsilon}\delta(x)dx} = \frac{1}{k}\int_{-\varepsilon}^{\varepsilon}\delta(x)dx$$

$$= \int_{-\varepsilon}^{\varepsilon}\frac{1}{k}\delta(x)dx$$

$$\therefore \quad \delta(kx) = \frac{1}{k}\delta(x) = \frac{1}{|k|}\delta(x) \qquad 1$$

Note for later that if $k>0$, then $k=|k|$

Case 2: $k<0$

If $k<0$, then can write $k=-|k|$

& $-k=|k|$ 0.5

$$\therefore \int_{-\varepsilon}^{\varepsilon} \delta(kx)\,dx = \frac{1}{k}\int_{-k\varepsilon}^{k\varepsilon} \delta(u)\,du$$

$$= -\frac{1}{|k|}\int_{|k|\varepsilon}^{-|k|\varepsilon} \delta(u)\,du$$

Flip integration limits. $$= \frac{1}{|k|}\underbrace{\int_{-|k|\varepsilon}^{|k|\varepsilon} \delta(u)\,du}_{1=\int \delta(x)\,dx} = \frac{1}{|k|}\int_{-\varepsilon}^{\varepsilon} \delta(x)\,dx$$

$$= \int_{-\varepsilon}^{\varepsilon} \frac{1}{|k|}\delta(x)\,dx$$

1

$$\therefore\ \delta(kx) = \frac{1}{|k|}\delta(x).$$

$\therefore$ Both the $k>0$ & $k<0$ cases lead to

$$\boxed{\delta(kx) = \frac{1}{|k|}\delta(x)}$$

(b) $$I = \int_{-\infty}^{\infty} (5x+1)\,\delta[4(x-2)]\,dx$$

$$\delta[4(x-2)] = \frac{1}{4}\delta(x-2)$$

0.5

$$\therefore\ I = \frac{1}{4}\int_{-\infty}^{\infty} (5x+1)\,\delta(x-2)\,dx$$

delta fcn selects the value $x-2=0$ $\Rightarrow x=2$

$$= \frac{1}{4}(5x+1)\Big|_{x=2} = \boxed{\frac{11}{4}}$$

1

5. (a) Show that $\frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r\sin\theta}$

Start w/ $y = r\sin\theta\sin\phi$ & $x = r\sin\theta\cos\phi$

s.t. $\frac{y}{x} = \tan\phi$ 1

$$\therefore \frac{\partial}{\partial x}\left(\tan\phi\right) = \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$ 0.5

$$\Downarrow$$

$$\sec^2\phi \frac{\partial\phi}{\partial x} = -\frac{y}{x^2} = -\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta\cos^2\phi}$$

$$\therefore \cancel{\sec^2\phi}\frac{\partial\phi}{\partial x} = -\frac{\sin\phi\,\cancel{\sec^2\phi}}{r\sin\theta}$$

$$\boxed{\therefore \frac{\partial\phi}{\partial x} = \frac{-\sin\phi}{r\sin\theta}}$$ 1

(b) Know that:

$$\frac{\partial T}{\partial x} = \sin\theta\cos\phi\,\frac{\partial T}{\partial r} + \frac{\cos\theta\cos\phi}{r}\,\frac{\partial T}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\,\frac{\partial T}{\partial \phi}$$

$$\frac{\partial T}{\partial y} = \sin\theta\sin\phi\,\frac{\partial T}{\partial r} + \frac{\cos\theta\sin\phi}{r}\,\frac{\partial T}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\,\frac{\partial T}{\partial \phi}$$

$$\frac{\partial T}{\partial z} = \cos\theta\,\frac{\partial T}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial T}{\partial \theta}$$

$$\therefore \vec{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$= \left[\sin\theta\cos\phi\,\underline{\frac{\partial T}{\partial r}} + \frac{\cos\theta\cos\phi}{r}\,\underline{\frac{\partial T}{\partial \theta}} - \frac{\sin\phi}{r\sin\theta}\,\underline{\frac{\partial T}{\partial \phi}}\right]\hat{x}$$

① $$+ \left[\sin\theta\sin\phi\,\underline{\frac{\partial T}{\partial r}} + \frac{\cos\theta\sin\phi}{r}\,\underline{\frac{\partial T}{\partial \theta}} + \frac{\cos\phi}{r\sin\theta}\,\underline{\frac{\partial T}{\partial \phi}}\right]\hat{y}$$

$$+ \left[\cos\theta\,\underline{\frac{\partial T}{\partial r}} - \frac{\sin\theta}{r}\,\underline{\frac{\partial T}{\partial \theta}}\right]\hat{z}$$

$$\therefore \vec{\nabla} T = \underbrace{\left[\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \right]}_{\hat{r}} \frac{\partial T}{\partial r}$$

0.5

$$+ \underbrace{\left[\frac{\cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}}{r} \right]}_{\frac{\hat{\theta}}{r}} \frac{\partial T}{\partial \theta}$$

0.5

$$+ \underbrace{\left[\frac{-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}}{r\sin\theta} \right]}_{\frac{\hat{\phi}}{r\sin\theta}} \frac{\partial T}{\partial \phi}$$

0.5

$$\boxed{\therefore \vec{\nabla} T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}}$$